

Min-max level estimate for a singular quasilinear polyharmonic equation in \mathbb{R}^{2m}

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Abstract

Using the framework first presented by Ruf and Sani in [26], we give a proof of an Adams type inequality which can be applied to the functional

$$J_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^{2m}} \left(|\nabla^m u|^2 + \sum_{\gamma=0}^{m-1} a_\gamma(x) |\nabla^\gamma u|^2 \right) dx - \int_{\mathbb{R}^{2m}} \frac{F(x, u)}{|x|^\beta} dx - \epsilon \int_{\mathbb{R}^{2m}} h u dx.$$

Under two kinds of assumptions on the nonlinearity, we estimate the min-max level of the functional. As an application, a multiplicity result for the related singular quasilinear elliptic equation is proved.

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1. Introduction and main results

Let $\nabla^\gamma u$, $\gamma \in \{0, 1, 2, \dots, m\}$, be the γ -th order gradient of a function $u \in W^{m,2}(\mathbb{R}^{2m})$ which is defined by

$$\nabla^\gamma u := \begin{cases} \Delta^{\frac{\gamma}{2}} u & \gamma \text{ even,} \\ \nabla \Delta^{\frac{\gamma-1}{2}} u & \gamma \text{ odd.} \end{cases}$$

Here and throughout this paper, we use the notations that

$$\Delta^0 u = \nabla^0 u = u.$$

Consider the following nonlinear functional

$$J_\epsilon(u) = \frac{1}{2} \int_{\mathbb{R}^{2m}} \left(|\nabla^m u|^2 + \sum_{\gamma=0}^{m-1} a_\gamma(x) |\nabla^\gamma u|^2 \right) dx - \int_{\mathbb{R}^{2m}} \frac{F(x, u)}{|x|^\beta} dx - \epsilon \int_{\mathbb{R}^{2m}} h u dx \quad (1.1)$$

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which is related to the higher order partial differential equation

$$(-\Delta)^m u + \sum_{\gamma=0}^{m-1} (-1)^\gamma \nabla^\gamma \cdot (a_\gamma(x) \nabla^\gamma u) = \frac{f(x, u)}{|x|^\beta} + \epsilon h(x). \quad (1.2)$$

Here $m \geq 2$ is an even integer, ϵ is a small constant, the equation is defined on the whole Euclidean space of dimension $2m$, $0 \leq \beta < 2m$, $h(x) \not\equiv 0$ belongs to the dual space of E which will be defined later, $f(x, s) : \mathbb{R}^{2m} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function which satisfies some growth conditions and $a_\gamma(x)$ are continuous functions satisfying

(A₁) there exist positive constants a_γ , $\gamma = 0, 1, 2, \dots, m-1$, such that $a_\gamma(x) \geq a_\gamma$ for all $x \in \mathbb{R}^{2m}$;
(A₂) $(a_0(x))^{-1} \in L^1(\mathbb{R}^{2m})$.

This kind of equations has been extensively studied by many authors. When $m = 1$, for the case $\beta = 0$, the equation on a bounded domain Ω has been investigated in [4, 10, 11, 34]. The corresponding n -Laplacian problem on a bounded domain also appears in many contexts, for example, in [8, 23]. For an unbounded domain, the problem becomes different and for this case one can refer to [3, 7, 9] and the references therein. For the singular case, namely $0 < \beta < n$, one can refer to [5, 16, 30, 33] and the references therein. Due to the variational structure of this kind of equations, when $m = 1$, usually the existence of solutions is related to the Moser-Trudinger type inequality. The inequality was first established by Trudinger [28] and Moser [22] and it says that, for a bounded domain $\Omega \subset \mathbb{R}^n$ and any $0 \leq \alpha \leq \alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$,

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} e^{\alpha|u|^{\frac{n}{n-1}}} dx < \infty, \quad (1.3)$$

where ω_{n-1} is the area of the unit sphere in \mathbb{R}^n .

When $m \geq 2$, related results about the corresponding higher order equations on bounded domains can be found in [13, 15, 17, 24]. To deal with the higher order equations, we need a generalization of the Moser-Trudinger type inequality which is called the Adams type inequality. The classical Adams inequality given by Adams [2] reads, for a bounded domain $\Omega \subset \mathbb{R}^n$ and any $0 \leq \alpha \leq \alpha(m, n)$,

$$\sup_{u \in W_0^{m, \frac{n}{m}}(\Omega), \|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)} \leq 1} \int_{\Omega} e^{\alpha|u|^{\frac{n}{n-m}}} dx < \infty, \quad (1.4)$$

where

$$\alpha(m, n) = \begin{cases} \frac{n}{\omega_{n-1}} \left(\frac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \right)^{\frac{n}{n-m}} & m \text{ odd}, \\ \frac{n}{\omega_{n-1}} \left(\frac{\pi^{\frac{n}{2}} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right)^{\frac{n}{n-m}} & m \text{ even}. \end{cases}$$

After Adams' work, many authors extended the inequality on a bounded domain from different points of view, for example, one can see [6, 12, 27, 32]. In particular, we mention the following singular Adams type inequality on a bounded domain [19] which will be used later in our proof.

Theorem A *Let $0 \leq \beta < n$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then for any $0 \leq \alpha \leq (1 - \frac{\beta}{n})\alpha(m, n)$, we have*

$$\sup_{u \in W_0^{m, \frac{n}{m}}(\Omega), \|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)} \leq 1} \int_{\Omega} \frac{e^{\alpha|u|^{\frac{n}{n-m}}}}{|x|^\beta} dx < \infty. \quad (1.5)$$

Moreover, when m is an even number, the Sobolev space $W_0^{m, \frac{n}{m}}(\Omega)$ in the above supremum can be replaced by the Sobolev space $W_N^{m, \frac{n}{m}}(\Omega)$.

In Theorem A, $W_N^{m, \frac{n}{m}}(\Omega)$ is used to denote the space of functions with homogeneous Navier boundary conditions, namely,

$$W_N^{m, \frac{n}{m}}(\Omega) := \left\{ u \in W^{m, \frac{n}{m}}(\Omega) \mid \Delta^\gamma u|_{\partial\Omega} = 0 \text{ in the sense of traces for } 0 \leq \gamma < \frac{m}{2} \right\}.$$

By definition, we have $W_0^{m, \frac{n}{m}}(\Omega) \subset W_N^{m, \frac{n}{m}}(\Omega)$, thus $W_N^{m, \frac{n}{m}}(\Omega)$ is a larger Sobolev space.

It is easy to see that, when $\Omega \subseteq \mathbb{R}^n$ has infinite volume, the problem is that the integrals in both (1.3) and (1.4) become infinite and the inequalities do not make sense. For the Moser-Trudinger type inequality, this problem was solved in [7, 25] for dimension $n = 2$ and in [1, 20] for general dimension. Recently, for the Adams type inequality on an unbounded domain, Ruf and Sani [26] got the following result

Theorem B *Let m be an even integer less than n and $\phi(t) := e^t - \sum_{\gamma=0}^{\gamma_{\frac{n}{m}}-2} \frac{t^\gamma}{\gamma!}$, where $\gamma_{\frac{n}{m}} := \min \left\{ \gamma \in \mathbb{N} \mid \gamma \geq \frac{n}{m} \right\} \geq \frac{n}{m}$. There exists a constant $C_{m,n} > 0$ such that, for any domain $\Omega \subseteq \mathbb{R}^n$,*

$$\sup_{u \in W_0^{m, \frac{n}{m}}(\Omega), \|u\|_{m,n} \leq 1} \int_{\Omega} \phi \left(\alpha(m, n) |u|^{\frac{n}{n-m}} \right) dx \leq C_{m,n} \quad (1.6)$$

and this inequality is sharp.

Hereafter we use $\|u\|_{m,n}$ to denote the norm of u which is defined by

$$\|u\|_{m,n} := \|(-\Delta + I)^{\frac{m}{2}} u\|_{L^{\frac{n}{m}}},$$

where I denotes the identity operator.

After this, based on the ideas in Ruf and Sani's paper [26], there are several generalizations of this result from different points of view. Lam and Lu [18] improved Theorem B to the case that m is an odd integer. When $n = 2m$ and $m \geq 2$ is an even integer, $\|u\|_{m,n}$ becomes

$$\|u\|_{m,2m} := \|(-\Delta + I)^{\frac{m}{2}} u\|_{L^2}.$$

But to be more suitable to use when considering equation (1.2), it is better to establish a singular Adams type inequality using the norm

$$\|u\|_E^2 := \int_{\mathbb{R}^{2m}} \left(\sum_{\gamma=0}^m \tau_\gamma |\nabla^\gamma u|^2 \right) dx$$

instead of the norm $\|\cdot\|_{m,2m}$. Here $\tau_m = 1$ and $\tau_\gamma > 0$ for $\gamma = 0, 1, 2, \dots, m-1$. For the nonsingular case, namely $\beta = 0$, this was done in [29] for $n = 2m = 4$ and in [18] for general $n = 2m$. When $0 < \beta < n$, there are only results for the special dimension $n = 2m = 4$. In [29], Yang proved a result for the subcritical case $\alpha < \alpha(2, 4)$ and in [18], Lam and Lu generalized the result to the critical case $\alpha = \alpha(2, 4)$. In this paper, we consider the general case $n = 2m$ and get the following theorem

Theorem 1.1 Let $m \geq 2$ be an even integer, $\tau_m = 1$, $\tau_\gamma > 0$ for $\gamma = 0, 1, 2, \dots, m-1$ and $0 \leq \beta < 2m$, then for any $0 \leq \alpha \leq \left(1 - \frac{\beta}{2m}\right)\alpha(m, 2m)$,

$$\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \|u\|_E \leq 1} \int_{\mathbb{R}^{2m}} \frac{e^{\alpha u^2} - 1}{|x|^\beta} dx < \infty, \quad (1.7)$$

where $\alpha(m, 2m) = (4\pi)^m m!$. Furthermore, the inequality is sharp.

From now on we assume that $m \geq 2$ is an even integer and the dimension n of the domain satisfies $n = 2m$. Motivated by the Adams type inequality above, we assume the following growth condition on the nonlinearity $f(x, s)$ of equation (1.2).

(H₁) There exist constants $\alpha_0, b_1, b_2 > 0$ and $\theta \geq 1$ such that for all $(x, s) \in \mathbb{R}^{2m} \times \mathbb{R}$,

$$|f(x, s)| \leq b_1 |s| + b_2 |s|^\theta (e^{\alpha_0 s^2} - 1).$$

(H₂) There exists $\mu > 2$ such that for all $x \in \mathbb{R}^{2m}$ and $s \neq 0$,

$$0 < \mu F(x, s) \equiv \mu \int_0^s f(x, t) dt \leq s f(x, s).$$

(H₃) There exist constants $R_0, M_0 > 0$ such that for all $x \in \mathbb{R}^{2m}$ and $|s| \geq R_0$,

$$0 < F(x, x) \leq M_0 |f(x, s)|.$$

Define a function space

$$E := \left\{ u \in W^{m,2}(\mathbb{R}^{2m}) : \int_{\mathbb{R}^{2m}} (|\nabla^m u|^2 + \sum_{\gamma=0}^{m-1} a_\gamma(x) |\nabla^\gamma u|^2) dx < \infty \right\}$$

and denote the norm of $u \in E$ by

$$\|u\|_E := \left(\int_{\mathbb{R}^{2m}} (|\nabla^m u|^2 + \sum_{\gamma=0}^{m-1} a_\gamma(x) |\nabla^\gamma u|^2) dx \right)^{\frac{1}{2}}.$$

Here and in the sequel we use E^* to denote the dual space of E and assume $h(x) \in E^*$. Define a singular eigenvalue λ_β by

$$\lambda_\beta := \inf_{u \in E \setminus \{0\}} \frac{\|u\|_E^2}{\int_{\mathbb{R}^{2m}} \frac{u^2}{|x|^\beta} dx}. \quad (1.8)$$

Moreover, we assume

(H₄) $\limsup_{s \rightarrow 0} \frac{2|F(x, s)|}{s^2} < \lambda_\beta$ uniformly with respect to $x \in \mathbb{R}^{2m}$.

The functional J_ϵ satisfies the geometric conditions of the mountain-pass theorem. The proof is similar to those in [29] and [33]. Namely, there exist two constant $r_\epsilon > 0$ and $\vartheta_\epsilon > 0$ such that

$J_\epsilon(u) \geq \vartheta_\epsilon$ when $\|u\|_E = r_\epsilon$ and there exists some $e \in E$ satisfying $\|e\|_E > r_\epsilon$ such that $J_\epsilon(e) < 0$. Moreover, $J_\epsilon(0) = 0$. Then the min-max level C_M of J_ϵ is defined by

$$C_M = \min_{l \in \mathcal{L}} \max_{u \in l} J_\epsilon(u),$$

where $\mathcal{L} = \{l \in C([0, 1], E) : l(0) = 0, l(1) = e\}$. It is obvious that C_M has a lower bound ϑ_ϵ , namely $C_M \geq \vartheta_\epsilon$. We also want to give an explicit upper bound of C_M . To this end, we need the following additional assumptions

(H₅) $\liminf_{s \rightarrow +\infty} s f(x, s) e^{-\alpha_0 s^2} = +\infty$ uniformly with respect to $x \in \mathbb{R}^{2m}$

or

(H₅)' There exist constants $p > 2$ and C_p such that

$$|f(s)| \geq C_p |s|^{p-1},$$

where

$$C_p > \left(\frac{p-2}{p} \right)^{\frac{p-2}{2}} \left(\frac{\alpha_0}{\left(1 - \frac{\beta}{2m}\right) (4\pi)^m m!} \right)^{\frac{p-2}{2}} S_p^p, \\ S_p^p := \inf_{u \in E \setminus \{0\}} \frac{\|u\|_E}{\left(\int_{\mathbb{R}^{2m}} \frac{u^p}{|x|^\beta} dx \right)^{\frac{1}{p}}}. \quad (1.9)$$

Under each of these two assumptions, we can get the same estimate on the min-max level of (1.1). Precisely, we have

Theorem 1.2 *Assume either (H₅) or (H₅)', together with (H₂) and (H₃), then there exists $\epsilon_0 > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$, the min-max level C_M of (1.1) satisfies*

$$C_M < \left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m!}{2\alpha_0}. \quad (1.10)$$

We remark that the above two assumptions on $f(x, s)$ can not cover each other. For details, one can refer to [31, 33] for examples of $f(x, s)$ which can not satisfy these two assumptions simultaneously. As an application of the above estimate, we can get the following multiplicity result of equation (1.2). We can see later that the estimate on C_M plays a crucial role in the proof of Theorem 1.3.

Theorem 1.3 *Assume either (H₅) or (H₅)', together with (H₁) – (H₄), then there exists $\epsilon_1 > 0$ such that, for any $0 < \epsilon \leq \epsilon_1$, the equation (1.2) has at least two distinct weak solutions.*

We organize this paper as follows: In Section 2, we prove the Adams type inequality (Theorem 1.1). In Section 3, we estimate the min-max level of functional (1.1) (Theorem 1.2). As an application of these two theorems, we prove the multiplicity result in Section 4 (Theorem 1.3).

2. Adams type inequality

Before the proof of Theorem 1.1, we point out that for $\tau_\gamma \leq a_\gamma$, $\gamma = 0, 1, 2, \dots, m-1$,

$$\|u\|_{\tilde{E}} \leq \|u\|_E.$$

This is the reason why Theorem 1.1 can be used in the study of equation (1.2). Using the Sobolev norm

$$\|u\|_{W^{m,2}} := \left(\sum_{\gamma=0}^m \|\nabla^\gamma u\|_{L^2}^2 \right)^{1/2},$$

it is easy to see that the norm $\|\cdot\|_{\tilde{E}}$ is equivalent to the norm $\|\cdot\|_{W^{m,2}}$. Another fact worth to emphasize is the following lemma

Lemma 2.1. *Under assumptions (A_1) and (A_2) , we have that the space E is compactly embedded into the space $L^q(\mathbb{R}^{2m})$ for any $q \geq 1$.*

The proof of this lemma is essentially the same as the proof of Lemma 3.6 in [29]. But for the convenience of readers, we give a proof here.

Proof. When $q \geq 2$, it is easy to see that the embedding $E \hookrightarrow L^q(\mathbb{R}^{2m})$ is continuous. When $q = 1$, Hölder's inequality and (A_2) imply that

$$\int_{\mathbb{R}^{2m}} |u| dx \leq \left(\int_{\mathbb{R}^{2m}} \frac{1}{a_0(x)} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{2m}} a_0(x) u^2 dx \right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}^{2m}} \frac{1}{a_0(x)} dx \right)^{\frac{1}{2}} \|u\|_E.$$

When $1 < q < 2$, we have

$$\int_{\mathbb{R}^{2m}} |u|^q dx \leq \int_{\mathbb{R}^{2m}} (|u| + u^2) dx \leq \left(\int_{\mathbb{R}^{2m}} \frac{1}{a_0(x)} dx \right)^{\frac{1}{2}} \|u\|_E + \frac{1}{a_0} \|u\|_E^2.$$

Thus we have that, for any $q \geq 1$, the embedding $E \hookrightarrow L^q(\mathbb{R}^{2m})$ is continuous.

Next we prove that the embedding is also compact. Suppose $\{u_k\} \subset E$ is a bounded sequence, we need to prove that u_k converges to some $u \in E$ strongly in $L^q(\mathbb{R}^{2m})$ up to a subsequence for any $q \geq 1$.

(A_2) implies that, for any $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\int_{|x| > R_\epsilon} \frac{1}{a_0(x)} dx < \epsilon^2.$$

Since $\{u_k\}$ is a bounded sequence, up to subsequence, we can assume that u_k converges to some u strongly in $L^1(B_{R_\epsilon})$. When $q = 1$, we have

$$\begin{aligned} \int_{\mathbb{R}^{2m}} |u_k - u| dx &= \int_{|x| \leq R_\epsilon} |u_k - u| dx + \int_{|x| > R_\epsilon} |u_k - u| dx \\ &\leq \int_{|x| \leq R_\epsilon} |u_k - u| dx + \left(\int_{|x| > R_\epsilon} \frac{1}{a_0(x)} dx \right)^{\frac{1}{2}} \left(\int_{|x| > R_\epsilon} a_0(x) |u_k - u|^2 dx \right)^{\frac{1}{2}} \\ &\leq \int_{|x| \leq R_\epsilon} |u_k - u| dx + \epsilon \|u_k - u\|_E. \end{aligned} \tag{2.1}$$

Noticing that ϵ can be arbitrarily small, we get from (2.1) that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^{2m}} |u_k - u| dx = 0.$$

When $q > 1$, we have

$$\begin{aligned} \int_{\mathbb{R}^{2m}} |u_k - u|^q dx &\leq \left(\int_{\mathbb{R}^{2m}} |u_k - u| dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{2m}} |u_k - u|^{2q-1} dx \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{\mathbb{R}^{2m}} |u_k - u| dx \right)^{\frac{1}{2}} \rightarrow 0 \end{aligned} \quad (2.2)$$

as $k \rightarrow \infty$. Here we used the continuous embedding $E \hookrightarrow L^{2q-1}(\mathbb{R}^{2m})$. \square

We remark here that the singular eigenvalue λ_β defined in (1.8) and S_p defined in (1.9) are both positive constants for any $0 \leq \beta < 2m$. When $\beta = 0$, (A_1) gives us that $\lambda_0 \geq a_0 > 0$. When $0 < \beta < 2m$, we have

$$\begin{aligned} \int_{\mathbb{R}^{2m}} \frac{u^2}{|x|^\beta} dx &\leq \int_{|x|>1} u^2 dx + \left(\int_{|x|\leq 1} |u|^{2q} dx \right)^{\frac{1}{q}} \left(\int_{|x|\leq 1} \frac{1}{|x|^{\beta q'}} dx \right)^{\frac{1}{q'}} \\ &\leq C \|u\|_E^2, \end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $0 < \beta q' < 2m$. This implies that $\lambda_\beta \geq \frac{1}{C} > 0$. Similarly, we can prove $S_p > 0$.

To prove Theorem 1.1, we first give several definitions. Let B_R be an open ball centered at 0 with radius $R > 0$. If $u : B_R \rightarrow \mathbb{R}$ is a measurable function, the distribution function of u is defined by

$$\mu_u(t) := \mathcal{M}(\{x \in B_R \mid |u(x)| > t\}) \quad \forall t \geq 0,$$

where $\mathcal{M}(\cdot)$ denotes the Lebesgue measure of a set in \mathbb{R}^n . The decreasing rearrangement of u is defined by

$$u^*(s) := \inf\{t \geq 0 \mid \mu_u(t) < s\} \quad \forall s \in [0, \mathcal{M}(B_R)].$$

Finally, the spherically symmetric decreasing rearrangement of u is defined by

$$u^*(x) := u^*(\sigma_n |x|^n) \quad \forall x \in B_R,$$

where σ_n is the volume of the unit ball in \mathbb{R}^n .

Now we begin to prove Theorem 1.1 by using the framework of Ruf and Sani's work [26]. After [26], similar ideas were also used in [18], [19] and [29].

Proof of Theorem 1.1. For any $u \in W^{m,2}(\mathbb{R}^{2m})$ and $\tilde{\rho} > 0$, direct computations give that

$$\int_{\mathbb{R}^{2m}} \left((-\Delta + \tilde{\rho} I)^{\frac{m}{2}} u \right)^2 dx = \sum_{\gamma=0}^m C(m, \gamma) \tilde{\rho}^{m-\gamma} \int_{\mathbb{R}^{2m}} |\nabla^\gamma u|^2 dx, \quad (2.3)$$

where $C(m, \gamma) = \frac{m!}{\gamma!(m-\gamma)!}$. In particular, $C(m, m) = C(m, 0) = 1$ and $C(m, 1) = m$. Define $\rho_\gamma = \left(\frac{\tau_\gamma}{C(m, \gamma)} \right)^{\frac{1}{m-\gamma}}$, $\gamma = \{0, 1, 2, \dots, m\}$ and let $\rho = \min\{\rho_0, \rho_1, \dots, \rho_m\}$. Then (2.3) tells us that

$$\int_{\mathbb{R}^{2m}} \left((-\Delta + \rho I)^{\frac{m}{2}} u \right)^2 dx \leq \|u\|_E^2.$$

So if we can prove that

$$\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \int_{\mathbb{R}^{2m}} ((-\Delta + \rho I)^{\frac{m}{2}} u)^2 dx \leq 1} \int_{\mathbb{R}^{2m}} \frac{e^{\alpha u^2} - 1}{|x|^\beta} dx < \infty,$$

the inequality in Theorem 1.1 is proved immediately.

By density of $C_0^\infty(\mathbb{R}^{2m})$ in $W^{m,2}(\mathbb{R}^{2m})$, we can find a sequence $\{u_k\} \subset C_0^\infty(\mathbb{R}^{2m})$ such that $u_k \rightarrow u$ in $W^{m,2}(\mathbb{R}^{2m})$. Without loss of generality, we can assume that $\int_{\mathbb{R}^{2m}} ((-\Delta + \rho I)^{\frac{m}{2}} u_k)^2 dx = 1$, for otherwise we can use $\tilde{u}_k = \frac{u_k}{\left(\int_{\mathbb{R}^{2m}} ((-\Delta + \rho I)^{\frac{m}{2}} u_k)^2 dx\right)^{\frac{1}{2}}}$ instead of u_k .

Suppose, for a fixed k , $\text{supp } u_k \subset B_{R_k}$. Define

$$f_k = (-\Delta + \rho I)^{\frac{m}{2}} u_k$$

and use f_k^* to denote the spherically symmetric decreasing rearrangement of f_k . Consider the equation

$$\begin{cases} (-\Delta + \rho I)^{\frac{m}{2}} v_k = f_k^* & \text{in } B_{R_k}, \\ v_k \in W_N^{m,2}(B_{R_k}). \end{cases}$$

By properties of rearrangement (see [14, 19, 26]), we have

$$\int_{B_{R_k}} ((-\Delta + \rho I)^{\frac{m}{2}} v_k)^2 dx = \|f_k^*\|_{L^2(B_{R_k})}^2 = \|f_k\|_{L^2(B_{R_k})}^2 = \int_{B_{R_k}} ((-\Delta + \rho I)^{\frac{m}{2}} u_k)^2 dx = 1 \quad (2.4)$$

and

$$\int_{B_{R_k}} \frac{e^{\alpha v_k^2} - 1}{|x|^\beta} dx \leq \int_{B_{R_k}} \frac{e^{\alpha v_k^2} - 1}{|x|^\beta} dx. \quad (2.5)$$

Let $r_0 \geq 1$ be a constant to be determined later. If $R_k \leq r_0$, since

$$\|\nabla^m v_k\|_{L^2(B_{R_k})}^2 \leq \int_{B_{R_k}} ((-\Delta + \rho I)^{\frac{m}{2}} v_k)^2 dx = 1,$$

by Theorem A, we get

$$\int_{B_{R_k}} \frac{e^{\alpha v_k^2} - 1}{|x|^\beta} dx < C_{m,r_0},$$

where C_{m,r_0} is some constant depending on m and r_0 but not depending on k .

If $R_k > r_0$, rewrite $\int_{B_{R_k}} \frac{e^{\alpha v_k^2} - 1}{|x|^\beta} dx$ into

$$\int_{B_{r_0}} \frac{e^{\alpha v_k^2} - 1}{|x|^\beta} dx + \int_{B_{R_k} \setminus B_{r_0}} \frac{e^{\alpha v_k^2} - 1}{|x|^\beta} dx := I_1 + I_2.$$

Firstly, we estimate I_1 . Define, for $\gamma = \{1, 2, \dots, \frac{m}{2}\}$ and $x \in B_{r_0}$,

$$\xi_\gamma(|x|) = |x|^{m-2\gamma}.$$

Let

$$g_k(x) = \sum_{\gamma=1}^{\frac{m}{2}} d_{k,\gamma} \xi_\gamma(|x|),$$

where

$$d_{k,\gamma} = \frac{\Delta^{\frac{m}{2}-\gamma} v_k(r_0) - \sum_{\eta=1}^{\gamma-1} d_{k,\eta} \Delta^{\frac{m}{2}-\gamma} \xi_\eta(r_0)}{\Delta^{\frac{m}{2}-\gamma} \xi_\gamma(r_0)}.$$

Denote $(v_k(x) - g_k(x))$ by $\mu_k(x)$. By construction, we have (see Lemma 4.3 in [26]) $\nabla^m \mu_k = \nabla^m v_k$ in B_{r_0} , μ_k is a radial function with homogeneous Navier boundary conditions and for $0 < |x| \leq r_0$,

$$v_k^2(x) \leq \mu_k^2(x) \left(1 + C_m \sum_{\gamma=1}^{\frac{m}{2}} r_0^{1-4\gamma} \|\Delta^{\frac{m}{2}-\gamma} v_k\|_{W^{1,2}(B_{r_0})}^2 \right)^2 + C_{m,r_0}. \quad (2.6)$$

Since $r_0 > 1$, (2.6) implies that

$$v_k^2(x) \leq \mu_k^2(x) \left(1 + C_m r_0^{-3} \sum_{\gamma=0}^{m-1} \|\nabla^\gamma v_k\|_{L^2(B_{r_0})}^2 \right)^2 + C_{m,r_0}. \quad (2.7)$$

Define

$$\tilde{\mu}_k(x) := \mu_k(x) \left(1 + C_m r_0^{-3} \sum_{\gamma=0}^{m-1} \|\nabla^\gamma v_k\|_{L^2(B_{r_0})}^2 \right)$$

and

$$C_{\min} := \min\{C(m, \gamma) \rho^{m-\gamma} \mid 0 \leq \gamma \leq m-1\}.$$

Then we have

$$\begin{aligned} \|\nabla^m \tilde{\mu}_k\|_{L^2(B_{r_0})}^2 &= \left(1 + C_m r_0^{-3} \sum_{\gamma=0}^{m-1} \|\nabla^\gamma v_k\|_{L^2(B_{r_0})}^2 \right)^2 \|\nabla^m \mu_k\|_{L^2(B_{r_0})}^2 \\ &= \left(1 + C_m r_0^{-3} \sum_{\gamma=0}^{m-1} \|\nabla^\gamma v_k\|_{L^2(B_{r_0})}^2 \right)^2 \|\nabla^m v_k\|_{L^2(B_{r_0})}^2 \\ &= \left(1 + C_m r_0^{-3} \sum_{\gamma=0}^{m-1} \|\nabla^\gamma v_k\|_{L^2(B_{r_0})}^2 \right)^2 \left(1 - \sum_{\gamma=0}^{m-1} C(m, \gamma) \rho^{m-\gamma} \|\nabla^\gamma v_k\|_{L^2(B_{r_0})}^2 \right) \\ &\leq \left(1 + C_m r_0^{-3} \sum_{\gamma=0}^{m-1} \|\nabla^\gamma v_k\|_{L^2(B_{r_0})}^2 \right)^2 \left(1 - C_{\min} \sum_{\gamma=0}^{m-1} \|\nabla^\gamma v_k\|_{L^2(B_{r_0})}^2 \right), \end{aligned} \quad (2.8)$$

where we have used (2.3) and (2.4) at the third equality. Choose $r_0^3 \geq \max\{1, \frac{C_m}{C_{\min}}\}$, we get from (2.8) that

$$\|\nabla^m \tilde{\mu}_k\|_{L^2(B_{r_0})}^2 \leq 1. \quad (2.9)$$

Now by (2.7), we have

$$\begin{aligned}
I_1 &\leq \int_{B_{r_0}} \frac{e^{\alpha \mu_k^2(x) \left(1 + C_m r_0^{-3} \sum_{\gamma=0}^{m-1} \|\nabla^\gamma v_k\|_{L^2(B_{r_0})}^2\right) + \alpha C_m r_0} - 1}{|x|^\beta} dx \\
&= \int_{B_{r_0}} \frac{e^{\alpha \tilde{\mu}_k^2(x) + \alpha C_m r_0} - 1}{|x|^\beta} dx \\
&\leq e^{\alpha C_m r_0} \int_{B_{r_0}} \frac{e^{\alpha \tilde{\mu}_k^2(x)}}{|x|^\beta} dx.
\end{aligned}$$

Then (2.9) and Theorem A imply that

$$I_1 \leq C_{m, r_0, \alpha, \rho}. \quad (2.10)$$

Secondly, we deal with I_2 . The radial lemma (see Chapter 6 in [14]) tells us that for $v_k \in W_N^{m,2}(B_{R_k}) \subset W^{1,2}(\mathbb{R}^{2m})$, we have

$$|v_k(x)| \leq \sqrt{\frac{(m-1)!}{\pi^m}} |x|^{\frac{1}{2}-m} \|v_k\|_{W^{1,2}(\mathbb{R}^{2m})} \quad \text{a.e. in } \mathbb{R}^{2m}. \quad (2.11)$$

Take $r_0^{2m-1} \geq \frac{(m-1)!}{\pi^m} \frac{1}{\min\{m\rho^{m-1}, \rho^m\}}$. If $|x| \geq r_0$, by (2.11), we have

$$|v_k(x)| \leq \sqrt{\min\{m\rho^{m-1}, \rho^m\}} \|v_k\|_{W^{1,2}(\mathbb{R}^{2m})}. \quad (2.12)$$

On the other hand, we have

$$\begin{aligned}
\min\{m\rho^{m-1}, \rho^m\} \|v_k\|_{W^{1,2}(\mathbb{R}^{2m})}^2 &\leq \int_{B_{R_k}} (m\rho^{m-1} |\nabla v_k|^2 + \rho^m v_k^2) dx \\
&\leq \int_{B_{R_k}} ((-\Delta + \rho I)^{\frac{m}{2}} v_k)^2 dx \\
&= 1.
\end{aligned} \quad (2.13)$$

Obviously, (2.12) and (2.13) imply that $|v_k(x)| \leq 1$ for any $|x| \geq r_0$. It follows that

$$\begin{aligned}
I_2 &\leq \frac{1}{r_0^\beta} \int_{B_{R_k} \setminus B_{r_0}} (e^{\alpha v_k^2} - 1) dx \\
&= \frac{1}{r_0^\beta} \int_{B_{R_k} \setminus B_{r_0}} \sum_{l=1}^{\infty} \frac{\alpha^l v_k^{2l}}{l!} dx \\
&\leq \frac{1}{r_0^\beta} \int_{B_{R_k} \setminus B_{r_0}} \sum_{l=1}^{\infty} \frac{\alpha^l v_k^2}{l!} dx \\
&\leq \frac{1}{r_0^\beta} \sum_{l=1}^{\infty} \frac{\alpha^l}{l!} \|v_k\|_{W^{1,2}(\mathbb{R}^{2m})}^2 \\
&\leq \frac{1}{\min\{m\rho^{m-1}, \rho^m\} r_0^\beta} \sum_{l=1}^{\infty} \frac{\alpha^l}{l!} \\
&\leq C_{m, r_0, \alpha, \rho}.
\end{aligned} \quad (2.14)$$

Take $r_0 \geq \max\{1, (\frac{C_m}{C_{\min}})^{\frac{1}{3}}, (\frac{(m-1)!}{\pi^m} \frac{1}{\min\{mp^{m-1}, \rho^m\}})^{\frac{1}{2m-1}}\}$. Then Fatou's lemma together with (2.5), (2.10) and (2.14) tells that

$$\int_{\mathbb{R}^{2m}} \frac{e^{\alpha u^2} - 1}{|x|^\beta} dx \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^{2m}} \frac{e^{\alpha u_k^2} - 1}{|x|^\beta} dx \leq C_{m, r_0, \alpha, \rho}$$

and the proof of the inequality is finished.

To prove the sharpness of the inequality, we need a sequence of test functions. For this reason, we postpone the proof of sharpness till the end of Section 3. \square

3. Min-max level

In this section, we estimate the min-max level of J_ϵ . Firstly, we define a sequence of functions $\tilde{\phi}_k(x)$ by

$$\tilde{\phi}_k(x) = \begin{cases} \sqrt{\frac{\log k}{2M}} + \frac{1}{\sqrt{2M \log k}} \sum_{\gamma=1}^{m-1} \frac{(1-k|x|^2)^\gamma}{\gamma} & |x| \in [0, \frac{1}{\sqrt{k}}), \\ -\sqrt{\frac{2}{M \log k}} \log |x| & |x| \in [\frac{1}{\sqrt{k}}, 1), \\ \zeta_k(x) & |x| \in [1, \infty) \end{cases}$$

where

$$M = \frac{(4\pi)^m (m-1)!}{2}, \zeta_k \in C_0^\infty(B_2(0)), \zeta_k|_{\partial B_1(0)} = \zeta_k|_{\partial B_2(0)} = 0.$$

Moreover, for $\gamma = \{1, 2, \dots, m-1\}$, $\frac{\partial^\gamma \zeta_k}{\partial r^\gamma}|_{\partial B_1(0)} = (-1)^\gamma (\gamma-1)! \sqrt{\frac{2}{M \log k}}$, $\frac{\partial^\gamma \zeta_k}{\partial r^\gamma}|_{\partial B_2(0)} = 0$ and ζ_k , $|\nabla^\gamma \zeta_k|$ and $|\nabla^m \zeta_k|$ are all $O\left(\frac{1}{\sqrt{\log k}}\right)$. Obviously, $\tilde{\phi}_k(x)$ are continuous functions defined on \mathbb{R}^{2m} with compact supports.

To estimate the $W^{m,2}$ norms of $\tilde{\phi}_k$, we need the following lemma.

Lemma 3.1. *For $\gamma = \{1, 2, \dots, m-1\}$, the γ -th order derivatives of $\tilde{\phi}_k$ with respect to r satisfy*

$$\lim_{r \rightarrow \frac{1}{\sqrt{k}}^-} \frac{\partial^\gamma \tilde{\phi}_k}{\partial r^\gamma} = \lim_{r \rightarrow \frac{1}{\sqrt{k}}^+} \frac{\partial^\gamma \tilde{\phi}_k}{\partial r^\gamma} \quad (3.1)$$

and

$$\lim_{r \rightarrow 1^-} \frac{\partial^\gamma \tilde{\phi}_k}{\partial r^\gamma} = \lim_{r \rightarrow 1^+} \frac{\partial^\gamma \tilde{\phi}_k}{\partial r^\gamma}, \quad (3.2)$$

where $r = |x|$.

Proof. Direct computations give that, when $\frac{1}{\sqrt{k}} \leq r < 1$, the γ -th order derivatives of $\tilde{\phi}_k$ with respect to r are

$$(-1)^\gamma (\gamma-1)! \sqrt{\frac{2}{M \log k}} r^{-\gamma}.$$

Combining this with our assumptions on ζ_k , we get (3.2).

To get (3.1), we consider the following functions of r

$$t_k(r) = -\frac{1}{\sqrt{2M \log k}} \log r.$$

The Taylor series of $t_k(r)$ at $\frac{1}{k}$ is

$$\begin{aligned} t_k(r) &= \sum_{\gamma=0}^{\infty} \frac{t_k^{(\gamma)}(\frac{1}{k})}{\gamma!} (r - \frac{1}{k})^\gamma \\ &= \sqrt{\frac{\log k}{2M}} + \frac{1}{\sqrt{2M \log k}} \sum_{\gamma=1}^{\infty} \frac{(1 - kr)^\gamma}{\gamma}. \end{aligned}$$

We use $\tilde{t}_k(r)$ to denote the summation of the first m terms of the series, namely,

$$\tilde{t}_k(r) = \sqrt{\frac{\log k}{2M}} + \frac{1}{\sqrt{2M \log k}} \sum_{\gamma=1}^{m-1} \frac{(1 - kr)^\gamma}{\gamma}.$$

It is easy to know that, at $r = \frac{1}{k}$, for $\gamma = \{1, 2, \dots, m-1\}$, the γ -th order derivatives of $t_k(r)$ equal to those of $\tilde{t}_k(r)$ respectively. By the definitions of $\tilde{\phi}_k$, we have

$$\tilde{\phi}_k(x) = \begin{cases} \tilde{t}_k(r^2) & r \in [0, \frac{1}{\sqrt{k}}), \\ t_k(r^2) & r \in [\frac{1}{\sqrt{k}}, 1). \end{cases}$$

This fact implies (3.1) immediately. \square

We remark that to find the extremal of Adams inequality, Adams has constructed a sequence of functions in [2] which has properties similar to our sequence. But at first, Adams' functions have no explicit expressions. Moreover, our functions are defined on the whole space \mathbb{R}^{2m} instead of a bounded domain $\Omega \subset \mathbb{R}^{2m}$.

We claim that $\tilde{\phi}_k(x) \in W_0^{m,2}(\mathbb{R}^{2m})$ and, for $\gamma = \{0, 1, \dots, m-1\}$,

$$\|\nabla^\gamma \tilde{\phi}_k\|_{L^2}^2 = O\left(\frac{1}{\log k}\right),$$

while

$$\|\nabla^m \tilde{\phi}_k\|_{L^2}^2 = \|\Delta^{\frac{m}{2}} \tilde{\phi}_k\|_{L^2}^2 = 1 + O\left(\frac{1}{\log k}\right).$$

To prove the claim, we first point out that, by Lemma 3.1 and the formula for integration by parts, we can get the weak derivatives of $\tilde{\phi}_k(x)$ until order m by computations on each part of the domain. Therefore, we can estimate the $W^{m,2}$ norms of $\tilde{\phi}_k$ respectively.

Part I $\mathbb{R}^{2m} \setminus B_1(0)$.

By definitions, since $\zeta_k \in C_0^\infty(B_2(0))$, we have, on $\mathbb{R}^{2m} \setminus B_1(0)$,

$$\|\tilde{\phi}_k\|_{L^2}^2 = \|\zeta_k\|_{L^2}^2 = O\left(\frac{1}{\log k}\right) \quad (3.3)$$

and, for $\gamma = \{1, 2, \dots, m\}$,

$$\|\nabla^\gamma \tilde{\phi}_k\|_{L^2}^2 = \|\nabla^\gamma \zeta_k\|_{L^2}^2 = O\left(\frac{1}{\log k}\right). \quad (3.4)$$

Part II $B_1(0) \setminus B_{\frac{1}{\sqrt{k}}}(0)$.

When $\gamma = 1$, it is easy to get

$$|\nabla^\gamma \tilde{\phi}_k| = |\nabla \tilde{\phi}_k| = -\sqrt{\frac{2}{M \log k}} r^{-1}. \quad (3.5)$$

For higher order derivatives, noticing the fact that, for any integer $1 \leq l \leq \frac{m}{2}$,

$$\Delta^l \log r = (-1)^{l-1} 2^{2l-1} \frac{(l-1)!(m-1)!}{(m-l-1)!} r^{-2l},$$

we have, when γ is odd and $1 < \gamma < m$,

$$\begin{aligned} |\nabla^\gamma \tilde{\phi}_k| &= \left| \frac{\partial}{\partial r} (\Delta^{\frac{\gamma-1}{2}} \tilde{\phi}_k) \right| \\ &= \left| (-1)^{\frac{\gamma+1}{2}} 2^{\gamma-2} (\gamma-1) \sqrt{\frac{2}{M \log k}} \frac{(m-1)! \left(\frac{\gamma-3}{2}\right)!}{\left(m - \frac{\gamma+1}{2}\right)!} r^{-(\gamma+1)} \right| \\ &= \left| 2^{\gamma-2} (\gamma-1) \sqrt{\frac{2}{M \log k}} \frac{(m-1)! \left(\frac{\gamma-3}{2}\right)!}{\left(m - \frac{\gamma+1}{2}\right)!} r^{-\gamma} \right|. \end{aligned} \quad (3.6)$$

When γ is even and $2 \leq \gamma \leq m$,

$$\nabla^\gamma \tilde{\phi}_k = \Delta^{\frac{\gamma}{2}} \tilde{\phi}_k = (-1)^{\frac{\gamma}{2}} 2^{\gamma-1} \sqrt{\frac{2}{M \log k}} \frac{(m-1)! \left(\frac{\gamma}{2} - 1\right)!}{\left(m - \frac{\gamma}{2} - 1\right)!} r^{-\gamma}. \quad (3.7)$$

In particular, we have

$$\nabla^m \tilde{\phi}_k = \Delta^{\frac{m}{2}} \tilde{\phi}_k = (-1)^{\frac{m}{2}} 2^{m-1} \sqrt{\frac{2}{M \log k}} (m-1)! r^{-m},$$

which gives us that, on $B_1(0) \setminus B_{\frac{1}{\sqrt{k}}}(0)$,

$$\begin{aligned} \|\nabla^m \tilde{\phi}_k\|_{L^2}^2 &= \int_{B_1(0) \setminus B_{\frac{1}{\sqrt{k}}}(0)} \frac{2^{2m-1} ((m-1)!)^2}{M \log k} r^{-2m} dx \\ &= \frac{2^{2m-1} ((m-1)!)^2}{M \log k} \int_{\frac{1}{\sqrt{k}}}^1 \omega_{2m-1} r^{-1} dr \\ &= \frac{4^{m-1} ((m-1)!)^2 \omega_{2m-1}}{M}. \end{aligned}$$

Substituting $M = \frac{(4\pi)^\gamma (m-1)!}{2}$ and $\omega_{2m-1} = \frac{2\pi^m}{(m-1)!}$, we get

$$\|\nabla^m \tilde{\phi}_k\|_{L^2}^2 = 1. \quad (3.8)$$

When $\gamma = 0$, by integrating by parts and the definitions of $\tilde{\phi}_k$, we get

$$\begin{aligned} \|\tilde{\phi}_k\|_{L^2}^2 &= \int_{B_1(0) \setminus B_{\frac{1}{\sqrt{k}}}(0)} \frac{2 \log^2 r}{M \log k} dx \\ &= \frac{\omega_{2m-1}}{M \log k} \left(\frac{1}{2m^3} - \frac{\log^2 k}{4mk^m} - \frac{\log k}{2m^2 k^m} - \frac{1}{2m^3 k^m} \right) \\ &= O\left(\frac{1}{\log k}\right). \end{aligned} \quad (3.9)$$

Similarly, when $\gamma = 1$, (3.5) gives us that

$$\begin{aligned} \|\nabla \tilde{\phi}_k\|_{L^2}^2 &= \int_{B_1(0) \setminus B_{\frac{1}{\sqrt{k}}}(0)} \frac{2r^{-2}}{M \log k} dx \\ &= \frac{\omega_{2m-1}}{(m-1)M \log k} \left(1 - \frac{1}{k^{m-1}} \right) \\ &= O\left(\frac{1}{\log k}\right). \end{aligned} \quad (3.10)$$

When γ is odd and $1 < \gamma < m$, by (3.6), we have

$$\begin{aligned} \|\nabla^\gamma \tilde{\phi}_k\|_{L^2}^2 &= \int_{B_1(0) \setminus B_{\frac{1}{\sqrt{k}}}(0)} \frac{(\gamma-1)^2 2^{2\gamma-3}}{M \log k} \left(\frac{(m-1)! \left(\frac{\gamma-3}{2}\right)!}{\left(m - \frac{\gamma+1}{2}\right)!} \right)^2 r^{-2\gamma} dx \\ &= \frac{(\gamma-1)^2 4^{\gamma-2} \omega_{2m-1}}{(m-\gamma)M \log k} \left(\frac{(m-1)! \left(\frac{\gamma-3}{2}\right)!}{\left(m - \frac{\gamma+1}{2}\right)!} \right)^2 \left(1 - \frac{1}{k^{m-\gamma}} \right) \\ &= O\left(\frac{1}{\log k}\right). \end{aligned} \quad (3.11)$$

When γ is even and $2 \leq \gamma \leq m-2$, by (3.7), we have

$$\begin{aligned} \|\nabla^\gamma \tilde{\phi}_k\|_{L^2}^2 &= \int_{B_1(0) \setminus B_{\frac{1}{\sqrt{k}}}(0)} \frac{2^{2\gamma-1}}{M \log k} \left(\frac{(m-1)! \left(\frac{\gamma}{2}-1\right)!}{\left(m - \frac{\gamma}{2}-1\right)!} \right)^2 r^{-2\gamma} dx \\ &= \frac{4^{\gamma-1} \omega_{2m-1}}{M \log k (m-\gamma)} \left(\frac{(m-1)! \left(\frac{\gamma}{2}-1\right)!}{\left(m - \frac{\gamma}{2}-1\right)!} \right)^2 \left(1 - \frac{1}{k^{m-\gamma}} \right) \\ &= O\left(\frac{1}{\log k}\right). \end{aligned} \quad (3.12)$$

Part III $B_{\frac{1}{\sqrt{k}}}(0)$.

We have

$$\|\tilde{\phi}_k\|_{W^{m,2}} \leq \left\| \sqrt{\frac{\log k}{2M}} \right\|_{W^{m,2}} + \sum_{\gamma=1}^{m-1} \left\| \frac{k^\gamma}{\gamma \sqrt{2M \log k}} \left(\frac{1}{k} - |x|^2\right)^\gamma \right\|_{W^{m,2}}.$$

Direct computations show that

$$\begin{aligned} \left\| \sqrt{\frac{\log k}{2M}} \right\|_{W^{m,2}(B_{\frac{1}{\sqrt{k}}})}^2 &= \int_{B_{\frac{1}{\sqrt{k}}}} \frac{\log k}{2M} dx \\ &= \frac{\omega_{2m-1} \log k}{4mMk^m} \\ &= o\left(\frac{1}{\log k}\right). \end{aligned} \quad (3.13)$$

Furthermore, by integrating by parts, we have

$$\begin{aligned} \left\| \frac{k^\gamma}{\gamma \sqrt{2M \log k}} \left(\frac{1}{k} - |x|^2\right)^\gamma \right\|_{W^{m,2}}^2 &= \sum_{\eta=0}^m \left\| \frac{k^\gamma}{\gamma \sqrt{2M \log k}} \nabla^\eta \left(\left(\frac{1}{k} - |x|^2\right)^\gamma\right) \right\|_{L^2(B_{\frac{1}{\sqrt{k}}}(0))}^2 \\ &= \begin{cases} \sum_{\eta=0}^{2\gamma} O\left(\frac{1}{k^{m-\eta} \log k}\right) & \gamma < \frac{m}{2}, \\ \sum_{\eta=0}^m O\left(\frac{1}{k^{m-\eta} \log k}\right) & \gamma \geq \frac{m}{2} \end{cases} \\ &= o\left(\frac{1}{\log k}\right). \end{aligned} \quad (3.14)$$

Combining (3.3), (3.4) and (3.8-3.14), we prove the claim that

$$\|\tilde{\phi}_k\|_{W^{m,2}(\mathbb{R}^{2m})}^2 = 1 + O\left(\frac{1}{\log k}\right),$$

or equivalently,

$$\|\tilde{\phi}_k\|_E^2 = 1 + O\left(\frac{1}{\log k}\right),$$

Define

$$\phi_k(x) = \frac{\tilde{\phi}_k(x)}{\|\tilde{\phi}_k\|_E}.$$

We have $\|\phi_k\|_E = 1$. Furthermore, we have that

$$\phi_k^2(x) \geq \frac{\log k}{2M} + O(1) \text{ for } |x| \leq \frac{1}{\sqrt{k}}. \quad (3.15)$$

Now we can begin the proof of Theorem 1.2.

Proof of Theorem 1.2. By (H_2) , we have $F(x, t\phi_k) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}^{2m}$. This implies that

$$\int_{\mathbb{R}^{2m}} \frac{F(x, t\phi_k)}{|x|^\beta} dx \geq \int_{|x| \leq \frac{1}{\sqrt{k}}} \frac{F(x, t\phi_k)}{|x|^\beta} dx.$$

By (3.15), we have, for t and k sufficiently large, there exists a constant $C_\phi > 0$ such that

$$t\phi_k > C_\phi \text{ for } |x| \leq \frac{1}{\sqrt{k}}.$$

Since (H_2) implies that, for $s > \frac{C_\phi}{2}$,

$$\int_{\frac{C_\phi}{2}}^s \frac{\mu}{t} dt \leq \int_{\frac{C_\phi}{2}}^s \frac{f(x, t)}{F(x, t)} dt,$$

we have, if t and k sufficiently large, for $|x| \leq \frac{1}{\sqrt{k}}$,

$$F(x, t\phi_k) \geq Ct^\mu \phi_k^\mu.$$

Therefore,

$$\begin{aligned} J_\epsilon(t\phi_k) &= \frac{t^2}{2} \int_{\mathbb{R}^{2m}} \left(|\nabla^m \phi_k|^2 + \sum_{\gamma=0}^{m-1} a_\gamma(x) |\nabla^\gamma \phi_k|^2 \right) dx - \int_{\mathbb{R}^{2m}} \frac{F(x, t\phi_k)}{|x|^\beta} dx - \epsilon t \int_{\mathbb{R}^{2m}} h\phi_k dx \\ &\leq \frac{t^2}{2} \int_{\mathbb{R}^{2m}} \left(|\nabla^m \phi_k|^2 + \sum_{\gamma=0}^{m-1} a_\gamma(x) |\nabla^\gamma \phi_k|^2 \right) dx - \int_{|x| \leq \frac{1}{\sqrt{k}}} \frac{F(x, t\phi_k)}{|x|^\beta} dx - \epsilon t \int_{\mathbb{R}^{2m}} h\phi_k dx \\ &\leq \frac{t^2}{2} \int_{\mathbb{R}^{2m}} \left(|\nabla^m \phi_k|^2 + \sum_{\gamma=0}^{m-1} a_\gamma(x) |\nabla^\gamma \phi_k|^2 \right) dx - Ct^\mu \int_{|x| \leq \frac{1}{\sqrt{k}}} \frac{\phi_k^\mu}{|x|^\beta} dx - \epsilon t \int_{\mathbb{R}^{2m}} h\phi_k dx \end{aligned}$$

Since $\mu > 2$, we get

$$\lim_{t \rightarrow +\infty} J_\epsilon(t\phi_k) = -\infty. \quad (3.16)$$

Suppose (1.10) is not correct. Then we have, for all k and $\epsilon > 0$,

$$\max_{t \geq 0} J_\epsilon(t\phi_k) \geq \left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m!}{2\alpha_0}. \quad (3.17)$$

(3.16) and (3.17) imply that, for any fixed k , there exists $t_k > 0$ such that

$$J_\epsilon(t_k\phi_k) = \max_{t \geq 0} J_\epsilon(t\phi_k).$$

It follows that $\frac{d}{dt} J_\epsilon(t\phi_k) = 0$ at $t = t_k$, or equivalently,

$$t_k^2 = t_k^2 \|\phi_k\|_E^2 = \int_{\mathbb{R}^{2m}} \frac{t_k \phi_k f(x, t_k \phi_k)}{|x|^\beta} dx + \epsilon t_k \int_{\mathbb{R}^{2m}} h\phi_k dx. \quad (3.18)$$

Now we claim that $\{t_k\}$ is a bounded sequence and its upper bound is independent of ϵ . Suppose not. (H_5) implies that, for any $\sigma > 0$, there exists $R_\sigma > 0$ such that, for all $s \geq R_\sigma$, it holds that

$$sf(x, s) \geq \sigma e^{\alpha_0 s^2}.$$

Then by (3.15) and (3.18), we have, for sufficiently large k ,

$$\begin{aligned}
t_k^2 &\geq \sigma \int_{|x| \leq \frac{1}{\sqrt{k}}} \frac{e^{\alpha_0 t_k^2 \phi_k^2}}{|x|^\beta} dx - \epsilon t_k \|h\|_{E^*} \|\phi_k\|_E \\
&\geq \sigma \int_{|x| \leq \frac{1}{\sqrt{k}}} \frac{e^{\alpha_0 t_k^2 \left(\frac{\log k}{2M} + O(1)\right)}}{|x|^\beta} dx - \epsilon t_k \|h\|_{E^*} \|\phi_k\|_E \\
&= \frac{\sigma \omega_{2m-1}}{(2m-\beta)k^{m-\frac{\beta}{2}}} e^{\alpha_0 t_k^2 \left(\frac{\log k}{2M} + O(1)\right)} - \epsilon t_k \|h\|_{E^*} \|\phi_k\|_E,
\end{aligned} \tag{3.19}$$

(3.19) is equivalent to

$$\begin{aligned}
1 &\geq \frac{\sigma \omega_{2m-1}}{(2m-\beta)} k^{\frac{\alpha_0 t_k^2}{2M} + o(1) - m + \frac{\beta}{2} - \frac{2 \log t_k}{\log k}} - \epsilon \|h\|_{E^*} \|\phi_k\|_E k^{\frac{\log t_k}{\log k}} \\
&\geq \frac{\sigma \omega_{2m-1}}{(2m-\beta)} k^{\frac{\alpha_0 t_k^2}{2M} + o(1) - m + \frac{\beta}{2} - \frac{2 \log t_k}{\log k}} - \|h\|_{E^*} \|\phi_k\|_E k^{\frac{\log t_k}{\log k}}.
\end{aligned}$$

Let $k \rightarrow \infty$, we get a contradiction because the right hand side of the inequality tends to $+\infty$. Thus the claim is proved.

From (3.17), we get that

$$t_k^2 \geq \left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m!}{\alpha_0} + 2\epsilon t_k \int_{\mathbb{R}^{2m}} h \phi_k dx. \tag{3.20}$$

Here we have used the fact that $\|\phi_k\|_E = 1$ and $F(x, s) \geq 0$. Since

$$\left| \epsilon t_k \int_{\mathbb{R}^{2m}} h \phi_k dx \right| \leq \epsilon t_k \|h\|_{E^*} \|\phi_k\|_E = \epsilon t_k \|h\|_{E^*},$$

t_k is bounded and ϵ can be arbitrarily small, (3.20) implies that

$$t_k^2 \geq \left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m!}{\alpha_0}.$$

If

$$\lim_{k \rightarrow \infty} t_k^2 > \left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m!}{\alpha_0},$$

we get, for sufficiently large k ,

$$\alpha_0 t_k^2 \frac{\log k}{2M} > \left(m - \frac{\beta}{2}\right) \log k.$$

This is a contradiction with the fact that $\{t_k\}$ is a bounded sequence because the right hand side of (3.19) tends to $+\infty$ as $k \rightarrow +\infty$. Thus we have

$$\lim_{k \rightarrow \infty} t_k^2 = \left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m!}{\alpha_0}.$$

Let $k \rightarrow \infty$ and $\epsilon \rightarrow 0$ in (3.19), we obtain

$$\left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m!}{\alpha_0} \geq \frac{\sigma \omega_{2m-1}}{(2m-\beta)}.$$

This is a contradiction because σ can be chosen arbitrarily large. Thus Theorem 1.2 is proved under assumption (H_5) .

If $f(x, s)$ satisfies $(H_5)'$ instead of (H_5) . We can choose a bounded sequence of functions $\{u_k\} \subset E$ such that

$$\int_{\mathbb{R}^{2m}} \frac{|u_k|^p}{|x|^\beta} dx = 1 \quad \text{and} \quad \|u_k\|_E \rightarrow S_p.$$

Then by Lemma 2.1, we can assume that there exists a function u_p such that

$$\begin{aligned} u_k &\rightharpoonup u_p && \text{in } E, \\ u_k &\rightarrow u_p && \text{in } L^q(\mathbb{R}^{2m}) \text{ for all } q \in [1, +\infty), \\ u_k(x) &\rightarrow u_p(x) && \text{almost everywhere.} \end{aligned}$$

These imply that

$$\int_{\mathbb{R}^{2m}} \frac{|u_k|^p}{|x|^\beta} dx \rightarrow \int_{\mathbb{R}^{2m}} \frac{|u_p|^p}{|x|^\beta} dx = 1.$$

On the other hand, we have

$$\|u_p\|_E \leq \liminf_{k \rightarrow \infty} \|u_k\|_E = S_p.$$

Thus we get $\|u_p\|_E = S_p$. Define a function $M_\epsilon(t) : [0, +\infty) \rightarrow \mathbb{R}$ by

$$M_\epsilon(t) := \frac{t^2}{2} \int_{\mathbb{R}^{2m}} \left(|\nabla^m u_p|^2 + \sum_{\gamma=0}^{m-1} a_\gamma(x) |\nabla^\gamma u_p|^2 \right) dx - \int_{\mathbb{R}^{2m}} \frac{F(x, tu_p)}{|x|^\beta} dx - t \epsilon \int_{\mathbb{R}^{2m}} h u_p dx$$

By (H_2) , $(H_5)'$ and $\int_{\mathbb{R}^{2m}} \frac{|u_p|^p}{|x|^\beta} dx = 1$, we have

$$\begin{aligned} M_\epsilon(t) &\leq \frac{t^2}{2} \int_{\mathbb{R}^{2m}} \left(|\nabla^m u_p|^2 + \sum_{i=0}^{m-1} a_i(x) |\nabla^i u_p|^2 \right) dx - C_p \frac{t^p}{p} \int_{\mathbb{R}^{2m}} \frac{|u_p|^p}{|x|^\beta} dx + \epsilon t \|h\|_{E^*} \|u_p\| \\ &= \frac{t^2}{2} S_p^2 - C_p \frac{t^p}{p} + \epsilon t S_p \|h\|_{E^*} \\ &\leq \frac{(p-2)}{2p} \frac{S_p^{2p/(p-2)}}{C_p^{2/(p-2)}} + \epsilon t_0 S_p \|h\|_{E^*}, \end{aligned}$$

where t_0 is a constant which belongs to $[0, +\infty)$ and is independent of the choice of ϵ . By choosing ϵ small enough, we get the desired results from the definitions of C_p and S_p in $(H_5)'$ immediately. \square

The sharpness of the Adams inequality.

Define $\varphi_k = \frac{\tilde{\phi}_k}{\|\tilde{\phi}_k\|_E}$. Obviously, we have

$$\begin{aligned}
\sup_{u \in W^{m,2}(\mathbb{R}^{2m}), \|u\|_E \leq 1} \int_{\mathbb{R}^{2m}} \frac{e^{\alpha u^2} - 1}{|x|^\beta} dx &\geq \int_{\mathbb{R}^{2m}} \frac{e^{\alpha \varphi_k^2} - 1}{|x|^\beta} dx \\
&\geq \int_{|x| \leq \frac{1}{\sqrt{k}}} \frac{e^{\alpha \varphi_k^2} - 1}{|x|^\beta} dx \\
&\geq \int_{|x| \leq \frac{1}{\sqrt{k}}} (Ck^{\frac{\alpha}{2M} + \frac{\beta}{2}} - k^{\frac{\beta}{2}}) dx \\
&= \frac{\omega_{2m-1} (Ck^{\frac{\alpha}{2M} + \frac{\beta}{2}} - k^{\frac{\beta}{2}})}{2mk^m}. \tag{3.21}
\end{aligned}$$

When $\alpha > \left(1 - \frac{\beta}{2m}\right) \alpha(m, 2m)$, substituting $M = \frac{(4\pi)^m(m-1)!}{2}$, we get

$$\frac{\alpha}{2M} + \frac{\beta}{2} > m.$$

This implies that the right hand side of (3.21) tends to infinity as $k \rightarrow \infty$. Thus the inequality (1.7) is sharp.

4. Multiplicity result of the related elliptic equation

To deal with equation (1.2), the main difference between our general case and the special case $n = 2m = 4$ is the function space E . As our proof of Lemma 2.1, the proofs of the following three lemmas are essentially the same as those in [29]. The different definitions of E do not cause difficulties and so we omit the proofs here.

Lemma 4.1. *Assume (A_1) , (A_2) and $(H_1) - (H_3)$. Then for any Palais-Smale sequence $\{u_k\} \subset E$ of J_ϵ , i.e.,*

$$J_\epsilon(u_k) \rightarrow C, J'_\epsilon(u_k) \rightarrow 0 \text{ as } k \rightarrow \infty,$$

up to subsequence, there exists $u \in E$ such that

$$u_k \rightharpoonup u \text{ in } E \text{ and } u_k \rightarrow u \text{ in } L^q(\mathbb{R}^{2m}) \text{ for any } q \geq 1.$$

Furthermore, we have

$$\begin{cases} \frac{f(x, u_k)}{|x|^\beta} \rightarrow \frac{f(x, u)}{|x|^\beta} \text{ in } L^1(\mathbb{R}^{2m}), \\ \frac{F(x, u_k)}{|x|^\beta} \rightarrow \frac{F(x, u)}{|x|^\beta} \text{ in } L^1(\mathbb{R}^{2m}) \end{cases}$$

and u is a weak solution of (1.2).

Lemma 4.2. *Assume (A_1) , (A_2) , (H_1) , (H_2) and (H_4) . Then there exists $\epsilon_2 > 0$ such that, for any $0 < \epsilon < \epsilon_2$, there exists a Palais-Smale sequence $\{u_k\} \subset E$ at level $C_0 < 0$ which converges strongly in E to a minimum type solution u_0 of (1.2). Furthermore, we have $C_0 \rightarrow 0$ as $\epsilon \rightarrow 0$.*

Lemma 4.3. *Assume (H_5) or $(H_5)'$, together with (A_1) , (A_2) and $(H_1) - (H_4)$. Then there exists $\epsilon_3 > 0$ such that, for any $0 < \epsilon < \epsilon_3$, there exists a Palais-Smale sequence $\{v_k\} \subset E$ at level*

$C_M > 0$ which converges weakly in E to a mountain-pass type solution v_0 of (1.2) with min-max level C_M .

By Lemma 4.2 and 4.3, to prove Theorem 1.3, we only need to prove that u_0 and v_0 are distinct weak solutions. During the proof, we need the following singular version of Lions' inequality. This kind of inequality was first proved by Lions in [21].

Lemma 4.4. *Let $\{w_k\}$ be a sequence in E . Suppose $\|w_k\|_E = 1$ and $w_k \rightharpoonup w_0$ in E . Then, for any $0 < p < \left(1 - \frac{\beta}{2m}\right) \frac{\alpha(m, 2m)}{1 - \|w_0\|_E^2}$, we have*

$$\sup_k \int_{\mathbb{R}^{2m}} \frac{e^{pw_k^2} - 1}{|x|^\beta} dx < \infty.$$

Proof of Lemma 4.4. If $w_0 \equiv 0$, the lemma is a direct consequence of Theorem 1.1. Otherwise, by our assumptions on w_k , we have

$$\begin{aligned} \|w_k - w_0\|_E^2 &= 1 + \|w_0\|_E^2 - 2 \int_{\mathbb{R}^{2m}} \left(\nabla^m w_k \nabla^m w_0 + \sum_{\gamma=0}^{m-1} a_\gamma(x) \nabla^\gamma w_k \nabla^\gamma w_0 \right) dx \\ &\rightarrow 1 - \|w_0\|_E^2 \text{ as } k \rightarrow \infty. \end{aligned} \quad (4.1)$$

Using Young's inequality, we have, for any $\delta > 0$,

$$\begin{aligned} \int_{\mathbb{R}^{2m}} \frac{e^{pw_k^2} - 1}{|x|^\beta} dx &\leq \int_{\mathbb{R}^{2m}} \frac{e^{p((1+\delta)(w_k - w_0)^2 + (1+\delta^{-1})w_0^2)} - 1}{|x|^\beta} dx \\ &\leq \frac{1}{\mu} \int_{\mathbb{R}^{2m}} \frac{e^{\mu p(1+\delta)(w_k - w_0)^2 - 1}}{|x|^\beta} dx + \frac{1}{\nu} \int_{\mathbb{R}^{2m}} \frac{e^{\nu p(1+\delta^{-1})w_0^2 - 1}}{|x|^\beta} dx \\ &=: W_1 + W_2. \end{aligned} \quad (4.2)$$

where $\mu > 1$, $\nu > 1$ and $\frac{1}{\mu} + \frac{1}{\nu} = 1$.

We can choose δ sufficiently small and μ sufficiently close to 1 such that

$$\mu p(1 + \delta)(1 - \|w_0\|_E^2) < \left(1 - \frac{\beta}{2m}\right) \alpha(m, 2m).$$

Then, by Theorem 1.1 and (4.1), we have $W_1 < C$ for some universal constant C .

To estimate W_2 , we first claim that, for any $\alpha > 0$ and $u \in E$, we have

$$\int_{\mathbb{R}^{2m}} \frac{e^{\alpha u^2} - 1}{|x|^\beta} dx < \infty.$$

In fact, since $E \subseteq W^{m,2}(\mathbb{R}^{2m})$, by the density of $C_0^\infty(\mathbb{R}^{2m})$ in $W^{m,2}(\mathbb{R}^{2m})$, we can choose some $u_0 \in C_0^\infty(\mathbb{R}^{2m})$ such that

$$\|u - u_0\|_{W^{m,2}(\mathbb{R}^{2m})}^2 < \left(1 - \frac{\beta}{2m}\right) \frac{\alpha(m, 2m)}{2\alpha}. \quad (4.3)$$

We get from (4.3) that

$$\left\| \sqrt{\left(\frac{2m}{2m-\beta}\right) \frac{2\alpha}{\alpha(m, 2m)} (u - u_0)} \right\|_{W^{m,2}(\mathbb{R}^{2m})} < 1. \quad (4.4)$$

Assume R_u and C_u are positive constants such that $\text{supp} u_0 \subset B_{R_u}$ and $|u_0| \leq C_u$. Then we have

$$\begin{aligned} \int_{\mathbb{R}^{2m}} \frac{e^{\alpha u^2} - 1}{|x|^\beta} dx &\leq \int_{\mathbb{R}^{2m}} \frac{e^{2\alpha(u-u_0)^2 + 2\alpha u_0^2} - 1}{|x|^\beta} dx \\ &= \int_{\mathbb{R}^{2m}} \frac{e^{2\alpha(u-u_0)^2 + 2\alpha u_0^2} - e^{2\alpha u_0^2} + e^{2\alpha u_0^2} - 1}{|x|^\beta} dx \\ &\leq e^{2\alpha C_u^2} \int_{\mathbb{R}^{2m}} \frac{e^{2\alpha(u-u_0)^2} - 1}{|x|^\beta} dx + \int_{B_{R_u}} \frac{e^{2\alpha u_0^2} - 1}{|x|^\beta} dx \\ &\leq e^{2\alpha C_u^2} \int_{\mathbb{R}^{2m}} \frac{e^{2\alpha(u-u_0)^2} - 1}{|x|^\beta} dx + (e^{2\alpha C_u^2} - 1) \int_{B_{R_u}} \frac{1}{|x|^\beta} dx \\ &= e^{2\alpha C_u^2} \int_{\mathbb{R}^{2m}} \frac{e^{2\alpha(u-u_0)^2} - 1}{|x|^\beta} dx + \frac{\omega_{2m-1}(e^{2\alpha C_u^2} - 1)}{2m - \beta} R_u^{2m-\beta} \end{aligned}$$

By (1.7) with $\tau_\gamma = 1$ for $0 \leq \gamma \leq m$ and (4.4), we have

$$\int_{\mathbb{R}^{2m}} \frac{e^{2\alpha(u-u_0)^2} - 1}{|x|^\beta} dx \leq C$$

for some universal constant C . Thus we get our claim proved and it is easy to see that $W_2 < C$ follows from the claim immediately. \square

Proof of Theorem 1.3. Suppose that u_0 and v_0 are the minimum and mountain-pass type solutions of (1.2) respectively. By Lemma 4.2 and 4.3, we have that, for ϵ small enough, there are two sequences $\{u_k\}$ and $\{v_k\}$ in E such that

$$\begin{aligned} u_k &\rightarrow u_0 \quad \text{and} \quad v_k \rightarrow v_0, \\ J_\epsilon(u_k) &\rightarrow C_0 < 0 \quad \text{and} \quad J_\epsilon(v_k) \rightarrow C_M > 0, \\ J'_\epsilon(u_k)u_k &\rightarrow 0 \quad \text{and} \quad J'_\epsilon(v_k)v_k \rightarrow 0. \end{aligned}$$

Theorem 1.2 tells us that $0 < C_M < C_0 + \left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m!}{2\alpha_0}$. We will show a contradiction under the assumption $u_0 = v_0$.

Let

$$w_k = \frac{v_k}{\|v_k\|_E} \quad \text{and} \quad w_0 = \frac{u_0}{\lim_{k \rightarrow \infty} \|v_k\|_E}.$$

We have $\|w_k\| = 1$ and $w_k \rightarrow w_0$ in E . In particular $\|w_0\|_E \leq 1$. To proceed, we distinguish two cases.

Case 1. $\|w_0\|_E = 1$.

In this case, we have

$$\lim_{k \rightarrow \infty} \|v_k\|_E = \|u_0\|_E.$$

Therefore, $v_k \rightarrow u_0$ in E . Lemma 4.1 tells us that

$$\frac{F(v_k)}{|x|^\beta} \rightarrow \frac{F(u_0)}{|x|^\beta} \quad \text{in } L^1(\mathbb{R}^{2m}).$$

Then we have

$$J_\epsilon(v_k) \rightarrow J_\epsilon(u_0) = C_0,$$

which is a contradiction with our assumption.

Case 2. $\|w_0\|_E < 1$.

Since $0 < C_M < C_0 + \left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m!}{2\alpha_0} = J_\epsilon(u_0) + \left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m!}{2\alpha_0}$, we can choose some $q > 1$ sufficiently close to 1 and $\delta > 0$ such that

$$q\alpha_0 \|v_k\|_E^2 \leq \left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m! \|v_k\|_E^2}{2(C_M - J_\epsilon(u_0))} - \delta.$$

Since $v_k \rightarrow u_0$ in E and $\frac{F(x, v_k)}{|x|^\beta} \rightarrow \frac{F(x, u_0)}{|x|^\beta}$ in $L^1(\mathbb{R}^{2m})$, we have

$$\lim_{k \rightarrow \infty} \|v_k\|_E^2 (1 - \|w_0\|_E^2) = \lim_{k \rightarrow \infty} \|v_k\|_E^2 - \|u_0\|_E^2 = 2(C_M - J_\epsilon(u_0)).$$

Then we get, for k sufficiently large,

$$q\alpha_0 \|v_k\|_E^2 < \left(1 - \frac{\beta}{2m}\right) \frac{(4\pi)^m m!}{1 - \|w_0\|_E^2}. \quad (4.5)$$

Suppose $\alpha > 0$, $p > 1$ and $p' > p$, using L'Hospital's rule, we have that there exists a positive constant C_α which only depends on α , such that for all $s > 0$,

$$(e^{\alpha s^2} - 1)^p \leq C_\alpha (e^{\alpha p' s^2} - 1).$$

In fact this is a result proved by the first author in [33]. Then by (H_1) and Hölder's inequality, we have

$$\begin{aligned} \int_{\mathbb{R}^{2m}} \frac{|f(x, v_k)|^p}{|x|^\beta} dx &\leq C \int_{\mathbb{R}^{2m}} \frac{|v_k|^p + |v_k|^{p\theta} (e^{\alpha_0 v_k^2} - 1)^p}{|x|^{p\beta}} dx \\ &\leq C \int_{\mathbb{R}^{2m}} \frac{|v_k|^p}{|x|^{p\beta}} dx + C \int_{\mathbb{R}^{2m}} \frac{|v_k|^{pp_1\theta}}{|x|^{p\beta}} dx \int_{\mathbb{R}^{2m}} \frac{(e^{\alpha_0 v_k^2} - 1)^{pp_2}}{|x|^{p\beta}} dx \\ &\leq C \int_{\mathbb{R}^{2m}} \frac{|v_k|^p}{|x|^{p\beta}} dx + C \int_{\mathbb{R}^{2m}} \frac{|v_k|^{pp_1\theta}}{|x|^{p\beta}} dx \int_{\mathbb{R}^{2m}} \frac{e^{\alpha_0 pp_2' v_k^2} - 1}{|x|^{p\beta}} dx, \end{aligned}$$

where $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and $p_2 < p_2'$. Since $0 \leq \beta < 2m$, (4.5) and the continuous embedding $E \hookrightarrow L^p(\mathbb{R}^{2m})$ for any $p \geq 1$ imply that there exists some $p : 1 < p < q$ such that $\frac{f(x, v_k)}{|x|^\beta}$ is bounded in $L^p(\mathbb{R}^{2m})$. It follows that

$$\left| \int_{\mathbb{R}^{2m}} \frac{f(x, v_k)(v_k - u_0)}{|x|^\beta} dx \right| \leq C \|v_k - u_0\|_{L^{\frac{p}{1-p}}} \rightarrow 0.$$

From this convergence and $J'_\epsilon(v_k)(v_k - u_0) \rightarrow 0$, we get

$$\int_{\mathbb{R}^{2m}} \left(\nabla^m v_k \nabla^m (v_k - u_0) + \sum_{\gamma=0}^{m-1} a_\gamma(x) \nabla^\gamma v_k \nabla^\gamma (v_k - u_0) \right) dx \rightarrow 0.$$

Moreover, since $v_k \rightharpoonup u_0$, we have

$$\int_{\mathbb{R}^{2m}} \left(\nabla^m u_0 \nabla^m (v_k - u_0) + \sum_{\gamma=0}^{m-1} a_\gamma(x) \nabla^\gamma u_0 \nabla^\gamma (v_k - u_0) \right) dx \rightarrow 0.$$

These two limitations tell us that $v_k \rightarrow u_0$ in E . From the continuity of the functional J_ϵ , we get $J_\epsilon(v_k) \rightarrow J_\epsilon(u_0) = C_0$, which is still a contradiction and the proof is finished. \square

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